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ON SOME RIGID SUBGROUPS OF SEMISIMPLE LIE GROUPS

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ABSTRACT

We show that certain discrete subgroups of semisimple Lie groups satisfy rigidity properties and that a subclass of these discrete groups are actually of finite covolume.

We show in this note that certain discrete subgroups of semisimple Lie groups behave like lattices i.e., are rigid subgroups, although it is not clear, *a priori*, that these are lattices. (See Theorem 1.) Using this rigidity, we show also that in many cases, these discrete subgroups are actually lattices (see Theorem 2).

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(0.1) DEFINITIONS AND NOTATIONS. Let Λ be a subgroup of a connected Lie group H. Suppose G' is an absolutely simple group defined over a local field k of characteristic 0.

Definition: A homomorphism $\rho\colon \Lambda \to G'(k)$ is **good** if $\rho(\Lambda)$ is Zariski dense in G' and if $\rho(\Lambda)$ is not contained in a compact subgroup of $G'(k)$.

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Definition: The subgroup Λ is **super-rigid** in H if every good homomorphism $\rho\colon \Lambda \to G'(k)$ extends to a continuous homomorphism $\widetilde{\rho}\colon H \to G'(k)$.

For example, (Theorem 2) of [Margulis 1] says that if Λ is an irreducible lattice in a real semisimple connected Lie group H of real rank ≥ 2 with no compact factors, then Λ is super-rigid in H.

THEOREM 1: *Let Δ be an irreducible lattice in a connected linear real semisimple Lie group G without compact factors and of real real rank ≥ 2. Let Γ be a Zariski dense discrete subgroup of $G \times G$ whose intersection with the diagonal subgroup G contains Δ. Then Γ is a super-rigid subgroup of $G \times G$.*

THEOREM 2: *Let Δ be an irreducible lattice in a cannected linear real semisimple Lie group G whose real rank ≥ 2 and which has no compact factors. Suppose G/Δ is not compact. Let $\Gamma \subset G \times G$ be a Zariski dense discrete subgroup whose intersection with the diagonal G contains Δ. Then the subgroup Γ is a lattice in $G \times G$.*

1. Rigidity

(1.1) Proof of Theorem 1: Let $\rho\colon \Gamma \to G'(k)$ be a good representation. Since $\rho(\Gamma)$ is not contained in a compact subgroup of $G'(k)$, it follows, by [3], that G' is isotropic over k, and by [2], Corollaire (4.17), G' contains a minimal parabolic k-subgroup P'. Let $\mathcal{P}$ denote the space of probability measures on the compact Hausdorff space $G'(k)/P'(k)$. Fix a minimal real parabolic subgroup P of G. Then by a lemma of Furstenberg (see [9], Chapter 2) there exists a Γ-equivariant measurable map $\varphi\colon G/P \times G/P \to \mathcal{P}$. Since Δ acts ergodically on $G/P \times G/P$ (see [9], Proposition (5.19)), it follows that Γ acts ergodically on $G/P \times G/P$. On the other hand, $G'(k)$ acts smoothly on $\mathcal{P}$ (see [9], Corollary (3.2.12)). It follows from well known considerations that the image of φ is contained in a $G'(k)$-orbit on $\mathcal{P}$. We get thus a closed subgroup H' of $G'(k)$ such that $\varphi\colon G/P \times G/P \to G'(k)/H'$.

We claim that k is archimedean. Indeed, suppose k is not archimedean. It is known that finite dimensional linear representations of Δ are completely reducible ([5], Ch. (7), Theorem (6.17)). It then follows from super rigidity of Δ in G (see Theorem 2 of [4]) that $\rho(\Delta)$ is contained in a compact subgroup C of $G'(k)$. Let $\{C_n\}$ be a sequence of compact open subgroups of C decreasing to $\{1\}$ and let

$\Delta_n = \rho^{-1}(C_n) \cap \Delta$. It is clear that Δ_n has finite index in Δ and is therefore a lattice in G and acts ergodically on $G/P \times G/P$. Being compact, C_n acts smoothly on $G'(k)/H'$. Therefore the image of φ is contained in an orbit O_n of C_n for every n, with $O_1 \supset O_2 \supset O_3 \supset O_4 \supset \cdots$. Since $\bigcap O_n$ is a point, we get $\mathbf{Im}(\varphi)$ is a point, i.e. $\mathbf{Im}(\varphi) = \mu$ for some $\mu \in \mathcal{P}$. Moreover, $\rho(\Gamma)$ is contained in the isotropy of μ. But Proposition (3.2.15) of [9] shows that the isotropy of μ is either a compact subgroup of $G'(k)$ or else is contained in a proper k-subgroup of $G'(k)$. Therefore $\rho(\Gamma)$ is either contained in a compact subgroup of $G'(k)$ or is contained in a proper k-subgroup of $G'(k)$, contradicting the hypothesis that $\rho\colon \Gamma \to G'(k)$ is good. Therefore k is archimedean.

Let G'' be the Zariski closure of $\rho(\Delta)$ in $G'(k)$. By replacing Δ by a subgroup of finite index, we may assume that $G''(k)$ is connected. Write $G''(k) = A.B$, where B is the maximal compact normal subgroup of $G''(k)$ and A is normal semisimple subgroup of $G''(k)$ without compact factors with $B \cap A$ being the center of A. The ergodicity of Δ-action on $G/P \times G/P$ implies that the image of φ is contained in an orbit O of $G''(k)$ in $G'(k)/H'$. Let $A\backslash O$ denote the quotient of O by $A, \pi\colon O \to A\backslash O$ the projection map, $\overline{\varphi}\colon G/P \times G/P \to A\backslash O$ the composite $\pi \circ \varphi$. The group $B/B \cap A$ acts on $A\backslash O$ and hence $\rho(\Delta)$ acts on $A\backslash O$ and $\overline{\varphi}\colon G/P \times G/P \to A\backslash O$ is Δ-equivariant.

We claim that $\overline{\varphi}$ is constant. Let S be a maximal split real torus in P, let P^0 be the parabolic subgroup of G containing S and opposed to P, let w be the element of the Weyl group $N_G(S)/Z_G(S)$ which conjugates P into P^0. The map $g \to (gP, gwP)$ into $G/P \times G/P$ maps $G/Z_G(S)$ isomorphically onto an open subset of $G/P \times G/P$ whose complement has zero measure. We may assume then that $\overline{\varphi}\colon G/Z_G(S) \to B/B_0$, where B_0 is a closed subgroup of B. We will show that $\overline{\varphi}$ coincide with homomorphisms on the centralizers of singular elements of S into B: let $\Phi(G, P, S)$ denote the root system of the triple (G, P, S), let Φ^+ be the set of positive roots, fix $\alpha \in \Phi^+$ and let $s \in S$ be an element such that $\alpha(s) = 1$ (*there exists such an* $s \in S$ *since* $\mathbf{R} - \mathrm{rank}(G) \geq 2$). For each $g \in G$, we get a map from $Z_G(s)$ into B/B_0 given by $\overline{\varphi}_g(z) = \overline{\varphi}(gz)$ for $z \in Z_G(s)$. We therefore get a map (where $\langle s \rangle$ denotes the group generated by S)

$$G/\langle s \rangle \quad \underset{(g \mapsto \overline{\varphi}_g)}{\longrightarrow} \quad \sum(Z(s), B/B_0)$$

where $\sum$ denotes the space of measurable maps from $Z(s)$ into B/B_0 (see [4], §3 for details). From the methods of [4], §3, it follows that $\overline{\varphi}_{gz} = h_g(z)\overline{\varphi}_g$ [where h_g is a continuous homomorphism from $Z(s)$ onto K, with K a quotient of a subgroup of B] for almost all $g \in G$ and $z \in Z_G(s)$. Let $u \in U_\alpha(k)$. Since $Z(s)$ is reductive ([2]) and $u \in U_\alpha \subset Z(s)$, it follows by the Jacobson–Morozov theorem that there exists a continuous homomorphism θ: $SL_2(\mathbf{R}) \to Z(s)$ such that $\theta \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u$. Now $h_g \circ \theta\,(SL_2(\mathbf{R}))$ is contained in the compact group K; using the fact (which is easy to prove) that there are no continuous homomorphisms from $SL_2(\mathbf{R})$ into a compact group K, we see that $h_g \circ \theta =$ trivial i.e. $h_g(u) = 1$ if $u \in U_\alpha$, with U_α denoting the unipotent subgroup of U corresponding to the root α. This shows that $\overline{\varphi}(gu) = \overline{\varphi}(g)$ for all $u \in U$ and similarly $\overline{\varphi}(gv) = \overline{\varphi}(g)$ for all $v \in U^-$ (U^- is the opposite of U) i.e., $\overline{\varphi}$ is constant.

We may then assume that φ: $G/P \times G/P \to \overline{A}/\overline{A}_0$ where $\overline{A} = A/Z(A)$ and $\overline{A}_0$ is a closed subgroup of $\overline{A}$. Since $\overline{A}$ is a product of absolutely simple groups, the super-rigidity of Δ in G implies that the homomorphism ρ_1: $\Delta \to G''(k)/B = \overline{A}$ extends to a homomorphism $\widetilde{\rho}_1$: $G \to \overline{A}$. The map φ: $G/Z_G(S) \to \overline{A}/\overline{A}_0$ is Δ-equivariant and thus the map $\widetilde{\varphi}$: $G \to \overline{A}/\overline{A}_0$ given by $\widetilde{\varphi}(g) = \widetilde{\rho}_1(g)^{-1}\varphi(g)$ is Δ-invariant on the left and $Z_G(S)$-equivariant on the right. Thus, $\widetilde{\varphi}$: $\Delta\backslash G \to \overline{A}/\overline{A}_0$ and $\mathbf{Im}(\widetilde{\varphi})$ is contained in an S-orbit in $\overline{A}/\overline{A}_0$ (since S acts ergodically on $\Delta\backslash G$ and smoothly on $\overline{A}/\overline{A}_0$), which has a finite S-invariant measure on it. But an S-orbit in $\overline{A}/\overline{A}_0$ is an algebraic group which is a split real torus, and since it has finite Haar-measure, it must be a point, and $\widetilde{\varphi}$ is constant: $\widetilde{\varphi}(g) = p$.

This shows that $\varphi(g) = \widetilde{\rho}_1(g)\widetilde{\varphi}(g) = \widetilde{\rho}(g)$ is a rational map from $G/P \times G/P \to G'(k)/H'$. From standard considerations (see [9], (5.1.3)) it follows that ρ: $\Gamma \to G'(k)$ extends to $\widetilde{\rho}_1$: $G \to G'(k)$. The proof of Theorem 1 is now complete.

(1.3) Notation and Remarks. Suppose Λ_1 and Λ_2 are subgroups of a group Λ. We say that $\Lambda_1 \prec \Lambda_2$ if $\Lambda_1/\Lambda_1 \cap \Lambda_2$ is finite. We write $\Lambda_1 \approx \Lambda_2$ if $\Lambda_1 \prec \Lambda_2$ and $\Lambda_2 \prec \Lambda_1$. We write $\Lambda_1 \precneqq \Lambda_2$ if $\Lambda_1 \prec \Lambda_2$ is true but $\Lambda_2 \prec \Lambda_1$ is not true.

The arithmeticity theorem (Theorem 1 of [9]) says the following: if Δ is an irreducible lattice in a connected linear semisimple group G without compact factors of real rank ≥ 2, then Δ is arithmetic, i.e., there exist (i) a semisimple algebraic group H defined over a number field K (ii) a surjection π: $H(K \otimes_{\mathbf{Q}} \mathbf{R}) \to G$ with compact kernel such that if O_K denotes the ring of integers in K and equalities are understood in the above sense, then $\Delta \approx \pi[H(O_K)]$.

Let S be the set of archimedean valuations v of K such that $H(K_\nu)$ is *non-*

compact. Then

$$G = \prod_{\nu \in S} H(K_\nu)$$

In fact the proof of arithmeticity from super-rigidity in [4] shows the following: *Let Δ, Γ be as in Theorem 1. Then Γ is contained in an arithmetic subgroup Λ of $G \times G$ since Γ is super-rigid. For details, see [7].*

(1.4) LEMMA: *Let G, H, K and π be as in (1.3). Let Λ be an arithmetic lattice in $G \times G$ which intersects the diagonal G in $\Delta \approx \pi[H(O_K)]$. Then Λ satisfies one of the following two conditions:*

(1) $\Lambda \approx \Delta \times \Delta$

(2) *there exists a real quadratic extension L of K (i.e. L such that every place of L lying over a real place of K is real) so that the map*

$$\pi \times \pi \colon H(L \otimes \mathbf{R}) = H(K \otimes \mathbf{R}) \times H(K \otimes \mathbf{R}) \to G \times G$$

has compact kernel with $\Lambda \approx (\pi \times \pi)[H(O_L)] \succ \pi[H(O_K)] \approx \Delta.$

Proof: By assumption, there exists a semisimple (centreless) group $\mathcal{G}$ defined over $\mathbf{Q}$ such that $\mathcal{G}(\mathbf{R}) = G \times G \times U$ where U is a compact group, and if $\mathrm{pr}\colon \mathcal{G}(\mathbf{R}) \to G \times G$ denotes the projection map, then $\Lambda \approx \mathrm{pr}[\mathcal{G}(\mathbf{Z})]$. Let $\mathcal{N}$ be the Zariski closure of $U \cap \mathcal{G}(\mathbf{Q})$ in $\mathcal{G}$. Then $\mathcal{N}$ is normal in $\mathcal{G}$ and $\mathcal{G}(\mathbf{R})/\mathcal{N}(\mathbf{R})$ is again of the form $G \times G \times V$ where V is a compact group, and if $\mathrm{pr}\colon \mathcal{G}(\mathbf{R})/\mathcal{N}(\mathbf{R}) \to G \times G$ denotes the projection map, then pr $[(\mathcal{G}/\mathcal{N})(\mathbf{Z})] \approx \Lambda$. We may therefore assume that $\mathcal{N}$ is trivial i.e. $U \cap \mathcal{G}(\mathbf{Q}) = \{1\}$. In that case, $\Lambda \approx \mathrm{pr}[\mathcal{G}(\mathbf{Z})]$ because $\mathcal{G}(\mathbf{Z}) \cap \ker(\mathrm{pr}) \subset \mathcal{G}(\mathbf{Q}) \cap U = \{1\}$.

Let $\mathcal{H}$ be the Zariski closure of $\mathrm{pr}^{-1}(\Delta) \cap \mathcal{G}(\mathbf{Z})$ in $\mathcal{G}$. Then $\mathrm{pr}(\mathcal{H}(\mathbf{R}))$ is the diagonal in $G \times G$ by the Borel density theorem applied to Δ in the diagonal G.

Using this, it is easy to show that $\mathcal{H}$ is a semisimple group. $\mathcal{H}$ is obviously defined over $\mathbf{Q}$. We have

$$\mathcal{H}(\mathbf{Z}) \approx \Delta \approx H(O_K) \text{ and } \mathcal{H}(\mathbf{Q}) \approx H(K) \text{ since}$$
$$\mathcal{H}(\mathbf{Q}) \cap \operatorname{Ker}(\mathrm{pr}) \subset \mathcal{G}(\mathbf{Q}) \cap U = \{1\}.$$

We have, as in (1.3), a finite set J and for $j \in J$, number fields L_j and an absolutely simple group $\mathcal{G}_j$ over L_j such that $\Lambda \approx \prod_j G_j(O_{L_j})$ and $\prod_j G_j(L_j \otimes \mathbf{R}) = \mathcal{G}(\mathbf{R})$. The fact that $U \cap \mathcal{G}(\mathbf{Q}) = \{1\}$ implies (since $G_j(O_{L_j})$

is finite if and only if $G_j(L_j \otimes \mathbf{R})$ is compact) that each $G_j(O_{L_j})$ is infinite. Moreover, $H(K) \approx \mathcal{H}(\mathbf{Q}) \subset \mathcal{G}(\mathbf{Q}) = \Pi_j G_j(L_j)$.

Since $\mathcal{G}_j(O_{Lj})$ is infinite, $\mathcal{G}_j(L_j \otimes_{\mathbf{Q}} \mathbf{R})$ is not compact. But the projection of Δ to any noncompact factor of $G \times G$ is nontrivial and therefore Δ projects nontrivially into $\mathcal{G}_j(O_{Lj})$ under the map $p_j\colon \mathcal{G}(\mathbf{Q}) \to \Pi_{\nu \in J} G_\nu(L_\nu) \to G_j(L_j)$.

This shows that $p_j\colon H(K) \simeq \mathcal{H}(\mathbf{Q}) \to G_j(L_j)$ is a nontrivial homomorphism induced by a morphism $\varphi_j\colon \mathcal{H} \to R_{L_j/\mathbf{Q}}(G_j)$ of $\mathbf{Q}$ groups. We note that $\mathcal{H}$ and G_j are absolutely simple groups. It follows that the morphism p_j is of the form $\rho_j \circ \sigma_j$ where $\sigma_j\colon K \to L_j$ is a field homomorphism and $\rho_j\colon \mathcal{H} \to G_j$ is a morphism of absolutely simple groups defined over L_j. Being non-trivial, ρ_j is an *isomorphism.* Thus we may assume that $L_j \supset K$ and $G_j = \mathbf{H} \otimes_K L_j$ for every j.

Let $\sum_j$ denote the set of archemedian places of L_j and for $w \in L_j$, let $L_{j,w}$ to the completion of L_j at w. Let $S_j = \{w \in \sum_j : G_j(L_{j,w}) \text{ is not compact }\}$. If S is as in (1.3), $v \in S$, $w \in \sum_j$ and w/v, then $G_j(L_{j,w}) = H(L_{j,w}) \supset H(K_\nu)$ and hence $w \in S_j$.

Then the equalities

$$G \times G \times U = \mathcal{G}(\mathbf{R}) = \prod_j G_j(L_j \otimes \mathbf{R}) = \prod_j \prod_{w \in \Sigma_j} G(L_{j,w})$$

show that

$$(*) \qquad G \times G = \prod_j \prod_{w \in S_j} G_j(L_{j,w}) \supset \prod_j \prod_{\nu \in S} H(L_j \otimes_K K_\nu)$$

holds. Let $\dim_{\mathbf{R}}(G)$ and $\dim(H)$ denote respectively, the dimension of G as a real group and the dimension of H as an algebraic group. Then from $(*)$ we obtain,

$$2 \dim_{\mathbf{R}}(G) = \dim_{\mathbf{R}}(G \times G) \geq \sum_j \sum_{\nu \in S} [L_j\colon K][K_\nu\colon \mathbf{R}] \dim\,(H)$$

and since $G = \prod_{\nu \in S} H(K_\nu)$, we obtain

$$2\ \dim_{\mathbf{R}}(G) \geq \sum_j [L_j\colon K] \dim_{\mathbf{R}}(G).$$

In particular card $(J) \leq 2$ and $[L_j\colon K] \leq 2$.

(1) If card $(J) = 2$, then $L_j = K$ for every j and $\Lambda \approx \Pi_{i \in J} H(O_{Lj}) \approx H(O_K) \times H(O_K)$.

(2) If card $(J) = 1$, $J = \{j\}$, then L_j/K is a quadratic extension and $\Lambda \approx H(O_{Lj})$. Now $(*)$ asserts the $(**)$ $G \times G = \Pi_{\nu \in S} H(L_j \otimes_K K_\nu)$.

By looking at the number of complex simple groups which are factors of either side of $(**)$, we obtain that L_j/K is a real quadratic extension. The Lemma is proved.

We show however, in the next section, that if G/Δ is not compact, then $\Gamma \approx \Lambda$ even in case (2).

2. Finiteness of volume

(2.1) PROPOSITION: *Let H be an absolutely almost simple group defined and isotropic over a number field K, let L/K be a real quadratic extension of K. Let Γ be a subgroup of $H(O_L)$ such that $\Gamma \cap H(O_K)$ has finite index in $H(O_K)$ and infinite index in Γ. Then Γ has finite index in $H(O_L)$.*

Proof: In the notation of (1.3), we must show that if $H(O_K) \underset{\neq}{\prec} \Gamma$ and $\Gamma \prec H(O_L)$, then $\Gamma = H(O_L)$. We note a few preliminary results.

(2.2) LEMMA: *Let Q be a minimal parabolic subgroup of H defined over K. Let V and V^- denote the unipotent radicals of Q and Q^-, where Q^- is a parabolic K-subgroup of H opposed to Q. Then the group generated by $V(NO_L)$ and $V^-(NO_L)$ has finite index in $H(O_L)$, where $V(NO_L) = \{v \in V(O_L); v \equiv 1 \pmod N\}$ and $V^-(NO_L)$ is defined similarly.*

For a proof, see [6] for K-rank $(H) \geq 2$ and [8] for K-rank $(H) = 1$.

Lemma (2.2) coupled with the Zariski density of Γ in H, shows that to prove (2.1), it is enough to show that $V(O_L) \prec \Gamma$.

(2.3) LEMMA: *Let S be a maximal K-split torus in Q, let $\Phi(H,Q,S)$ denote the root system of the triple (H,Q,S), let Φ^+ (resp. Δ) be the set of positive (resp. simple) roots, let $k \in [N_G(S)/Z_G(S)](K)$ be such that we have $k(\Phi^+) = -\Phi^+$ (then $k(\Delta) = -\Delta$). Let $S_0 = \{t \in S;$ we have $\alpha(t) = \beta(t)$ for all $\alpha, \beta \in \Delta\}$. Then*

(i) *S_0 is a one-dimensional split torus over K, normalized by k,*

(ii) *if $\varphi\colon \mathbf{G}_m \to S_0$ is an isomorphism suitably chosen, then $\alpha(\varphi(t)) = t$ for all $t \in \mathbf{G}_m$ and $\alpha \in \Delta$.*

The proof of this lemma is trivial and we omit it.

We now consider the group $R = S_0V$. Pick and element $\gamma \in \Gamma$ such that (i) $\gamma^{-1}\sigma(\gamma)$ and $\sigma(\gamma)^{-1}\gamma$ do not lie in a proper parabolic subgroup of H containing Q (ii) $\gamma^{-1}\sigma(\gamma)$ lies in the big Bruhat cell VkQ, where σ is a generator of Aut (L/K) which then acts on $H(L)$. For this γ, we can easily see using the Bruhat decomposition that $\gamma R\gamma^{-1} \cap \sigma(\gamma)R\sigma(\gamma)^{-1}$ is a one-dimensional torus T defined over K.

(2.4) LEMMA: *The torus T is anisotropic over K.*

Proof: Suppose T is isomorphic to $\mathbf{G}_m$ over K. Consider the conjugate of T by γ^{-1}: ${}^{\gamma^{-1}}(T)(\doteq \gamma^{-1}T\gamma) \subset R$. Now ${}^{\gamma^{-1}}(T)$ is a torus defined over L and contained in S_0V, and hence is conjugate to S_0 by an element $g \in (S_0V)(L)$. Thus ${}^{g^{-1}\gamma^{-1}}(T) = S_0 = \mathbf{G}_m$ over K by Lemma (2.3). The map $t \mapsto {}^{g^{-1}\gamma^{-1}}(t)(\doteq g^{-1}\gamma^{-1}t\gamma g)$ is a homomorphism of T into S_0, and $T = \mathbf{G}_m$ by assumption and $S_0 = \mathbf{G}_m$ by Lemma (2.3). But any homomorphism of $\mathbf{G}_m$ onto $\mathbf{G}_m$ is defined over K (in fact, over Q) which shows that for all $t \in T(K)$ we have

$$\begin{aligned} s &= {}^{g^{-1}\gamma^{-1}}(t) = \sigma(s) = {}^{\sigma(g)^{-1}\sigma(j)^{-1}}(\sigma(t)) = {}^{\sigma(g)^{-1}\sigma(\gamma)^{-1}}(t) \\ &= {}^{\sigma(g)^{-1}\sigma(\gamma)^{-1}\gamma g g^{-1}\gamma^{-1}}(t) = {}^{\sigma(g)^{-1}\sigma(\gamma)^{-1}\gamma g}(s). \end{aligned}$$

Therefore, $\sigma(g)^{-1}\sigma(\gamma)^{-1}\gamma g \in Z(S_0)$. By the choice of S_0 it is clear that $Z(S_0) \subset Q$; but $g, \sigma(g) \in R(L) \subset Q$. Therefore, $\sigma(\gamma)^{-1}\gamma \in QZ(S_0)Q \subset Q$ which contradicts the choice of γ. This proves Lemma (2.4).

(2.5) LEMMA: *Given any integer $N > 0$, let $T(NO_K) = \{t \in T(O_K); t \equiv 1 \,(\mathrm{mod}\, N)\}$. For $t \in T(NO_K)$ let*

$$\underline{\omega}(t) = \{w \in \underline{\nu};\ w = \gamma^{-1}t\gamma(v) - (\sigma(\gamma)^{-1}t\sigma(\gamma))(v) \text{ where } v \in \underline{\nu}\}.$$

Here $\underline{\nu}$ is the Lie algebra of V (defined over K) and ${}^{\gamma^{-1}t\gamma}(V)$ denotes the conjugate of V by $\gamma^{-1}t\gamma$. Then $\underline{\omega}(t)$ is a K-subspace of $\underline{\nu}$. Moreover, for a suitable choice $t, \underline{\omega}(t) = \underline{\nu}$.

Proof: It is clear that $\underline{\omega}(t)$ is a K-subspace of $\underline{\nu}$. Consider the map $\mathrm{pr}\colon R \to R/V = S_0$ and define the map $\psi_r\colon T \to S_0$ by

$$\psi_r(t) = \mathrm{pr}(\gamma^{-1}t^{-1}\gamma\sigma(\gamma)^{-1}t\sigma(\gamma)^{+1}).$$

This map cannot be trivial, because that would mean that the map $t \mapsto \mathrm{pr}(\gamma^{-1}t\gamma)$ from T onto S_0 is defined over K, but T, being anisotropic, cannot have nontrivial K-homomorphisms onto $S_0 = \mathbf{G}_m$.

CLAIM: *We can pick $t \in T(NO_K)$ of infinite order such that $\psi_\gamma(t) \neq \{1\}$.*

If N is divisible by sufficiently many primes, then $J(NO_K)$ is torsion free. Therefore, we need only prove that $T(NO_K)$ is a compact discrete subgroup of $T(K \otimes_{\mathbf{Q}} \mathbf{R})$, by (2.4) and the Godment criterion (see [1]). Since $T \approx \mathbf{G}_m$ over $\overline{K}$, we need only show that $T(O_K)$ is infinite.

Since S_0 and T are conjugate by $\gamma \in G(L)$, it follows that $S_0(L) \simeq T(\mathbf{L})$ and that T is split over $\mathbf{L}$, and is one-dimensional. It is thus clear that $T \subset R_{L/K}(\mathbf{G}_m)$. Since T is an isotropic over K by (2.4), we have: $T =$ Ker(Norm: $R_{L/K}(\mathbf{G}_m) \to \mathbf{G}_m$) where Norm is the Norm mapping of L over K.

Since L/K is a real extension,

$$\begin{aligned}[R_{L/K}(\mathbf{G}_m)](K \otimes \mathbf{R}) &= \mathbf{G}_m(L \otimes_{\mathbf{Q}} \mathbf{R}) \\ &= \prod_w (\mathbf{C}^* \times \mathbf{C}^*) \times \prod_v (\mathbf{R}^* \times \mathbf{R}^*),\end{aligned}$$

w a complex place of K and v a real place of K.

This shows that $T(K \otimes \mathbf{R})$ is noncompact, since

$$T(K \otimes \mathbf{R}) = \Pi_{v \text{ complex}}\, \mathbf{C}^* \times \Pi_{v \text{ real}} \mathbf{R}^*$$

where v runs through the archimedean places of K i.e. $T(O_K)$ is infinite and is hence Zarishi dense in T. The claim follows. Fix this t. Define M_0 as the K-subgroup generated by the element $\gamma^{-1}t^{-1}\gamma\sigma(\gamma)^{-1}t\sigma(\gamma)$ in $R = S_0V$. Then M_0 projects nontrivially onto S_0 under pr. We claim that $\underline{\omega}(t) = \underline{\nu}$ for this choice of t. Suppose not. Then there exist $v \in \underline{\nu} - \{0\}$ such that $\gamma^{-1}t\gamma(v) = \sigma(\gamma)^{-1}t\sigma(\gamma)(v)$ and therefore M_0 fixes the vector v. Since M_0 projects nontrivially onto S_0, there exists a torus in M_0 which is conjugate to S_0. Therefore S_0 fixes a conjugate of v in $\underline{\nu}$. But, by Lemma (2.3), S_0 does not fix any vector in $\underline{\nu} - \{0\}$. This shows that $\underline{\omega}(t) = \underline{\nu}$ and the lemma is proved.

We now complete the proof of Proposition (2.1).

CLAIM: *There exists an integer $C > 0$, $t \in T(CO_K)$ such that for the choice of γ as above, the following holds (the equalities are in the sense of (1.3)):*

$$\langle \exp\, [{}^{\gamma^{-1}t\gamma}\underline{\nu}(CO_K)],\ \exp\, [\underline{\nu}(CO_K)]\ \rangle \approx\ \exp\, [\underline{\nu}(CO_L)] \approx V(O_L)$$

where $\exp L$: $\underline{\nu} \to V$ *denotes the exponential map and* $\langle A, B\rangle$ *denote the group generated by* A *and* B.

On the other hand,

$$\Gamma \succ \langle \exp [{}^{\gamma^{-1}t\gamma}\underline{\nu}(CO_K)],\ \exp [\underline{\nu}(CO_K)]\rangle$$

since $\gamma \in \Gamma, t \in T(CO_K) \subset \Gamma$, $\exp [\underline{\nu}(CO_K)] = V(O_K) \prec \Gamma$. This shows that $\Gamma \succ V(O_L)$ and by Lemma (2.2) $\Gamma \approx H(O_L)$. We have thus proved Proposition (2.1).

Proof of Theorem 2: We have already seen, from Theorem 1 and (1.3), that if Γ is contained in an arithmetic lattice $\Lambda = \Delta \times \Delta$, then the comments in (1.4) show that $\Gamma = \Lambda$ and is a lattice. If, as in (1.4), we have $\Gamma \subset \pi \times \pi(H(O_L))$ where L/K is a real quadratic extension, then $\Delta = \pi[H(O_K)]$; since G/Δ is assumed to be non-compact, by [1] H is isotropic over K and therefore Theorem 1 applies, to show that $\Gamma = \Lambda$, i.e., Λ is indeed a lattice in $(G \times G)$. This completes the proof.

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